noneven. Its validity for the Klein-Gordon phase shift follows from the foregoing proof as well as from the relation[12] between the Dirac and Klein-Gordon phase shifts at high energy. It also follows in virtue of (4.22) that $\eta^{(1)}$ and, therefore, also the first-order phase shift, $\delta^{(1)}$, is a good approximation for the phase shift at high energy in case of nonsingular potentials. This is in agreement with the results of reference 7, where the higher order phase shifts were estimated by the WKB approximation. It was also shown there that for nonsingular, noneven potentials the first-order phase shift was sufficient to derive the leading term in the asymptotic expansion of the scattering amplitude. As far as the calculation of the amplitude for nonsingular potentials at high energy is concerned several questions remain unsettled: (i) Is the higher order WKB approximation of the phase shifts justified? (ii) Is $\eta^{(1)}$ also in the case of even potentials a good approximation of the phase shifts? In order to answer the first question satisfactorily a more detailed analysis of the asymptotic behavior of the phase shifts is necessary. This can be made by a more systematic use of integration by parts than made above. As for the second question we recall that the asymptotic amplitude for a noneven potential has been derived essentially from that part of the phase shift which depends on l in the form (see reference 7)

$$[\text{polynomial in } (l+\tfrac{1}{2})]\times\psi(l+1),$$

where ψ is the logarithmic derivative of the gamma function. On the other hand it has been shown in reference 7 that for an even potential the first-order phase shift does not depend on $\psi(l+1)$. From the first part of the last section of the present paper it follows that this is true to all orders. Therefore, to answer the second question one has to derive an amplitude from phase shifts which do not depend on $\psi(l+1)$.

ACKNOWLEDGMENTS

One of the authors (S. T.) would like to express his gratitude to Professor L. Spruch for useful remarks.

PHYSICAL REVIEW VOLUME 131, NUMBER 1 1 JULY 1963

Opening Angles of Electron-Positron Pairs

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(Received 11 January 1963)

The cross section for production of a high-energy electron-positron pair of opening angle θ and electron energy ϵ_1, $d^2\sigma(\theta,\epsilon_1)/d\theta d\epsilon_1$ is calculated. Comparison with available experimental data shows good agreement with the present theory. The cross section $d^2\sigma(\theta,\epsilon_1)/d\theta d\epsilon_1$ is shown to be closely related to the cross section for an angle θ_1 between the photon and the electron, $d^2\sigma(\theta_1,\epsilon_1)/d\theta_1 d\epsilon_1$. At high photon energies the functional dependence of $d^2\sigma(\theta,\epsilon_1)/d\theta d\epsilon_1$ on the variable $w=(\epsilon_1\epsilon_2/k)\theta$ is very nearly the same as the functional dependence of $d^2\sigma(\theta_1,\epsilon_1)/d\theta d\epsilon_1$ on the variable $u=\epsilon_1\theta_1$. The experimental method of estimating the energy of a photon creating a pair from the opening angle of the pair is discussed. Formulas for the most probable photon energy for a measured opening angle, including the effect of multiple scattering, are given.

1. INTRODUCTION

THE distribution of the opening angle between the electron and positron of pairs produced by gamma rays has been the subject of many experimental investigations.[1–11] In all cases it has been found that the experimental distribution is considerably more narrow than the theoretical distribution of Borsellino[12] to which the experimental results customarily have been compared. The solution to this puzzle is that Borsellino's cross section does not give the distribution of opening angles for a fixed value of the energy partition between the pair particles, but is rather the distribution function of a certain combination of opening angle and energy partition, viz., the invariant pair energy.

We calculate here the high-energy pair-production cross section as a function of opening angle and energy partition. The resulting distribution of the opening angle is found to be in good agreement with the experimental distributions.

The good agreement between theory and experiment gives one renewed confidence in the method in current use of estimating photon energies by measurement of the

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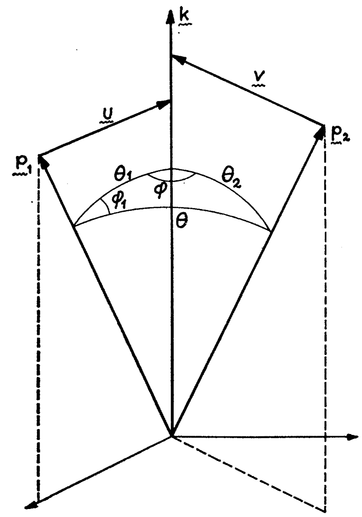

FIG. 1. Angles occurring in pair cross section.

opening angles of electron positron pairs. The method is discussed in some detail including the effect of multiple scattering.

2. DIFFERENTIAL PAIR CROSS SECTION

The differential high-energy pair-production cross section, summed over polarizations of electron and positron and averaged over photon polarization is in the Born approximation given by[13]

$$d\sigma=(\sigma_0/2\pi^2)I\frac{\epsilon_1^2\epsilon_2^2}{k^3}d\epsilon_1 d\Omega_1 d\Omega_2, \qquad (1)$$

with

$$\sigma_0=4\frac{e^2}{mc^2}\frac{\hbar}{mc}\left(\frac{Ze^2}{\hbar c}\right)^2$$

and

$$I=\frac{1}{q^4}\{(\epsilon_1^2+\epsilon_2^2)\xi\eta q_\perp^2+2\epsilon_1\epsilon_2(\xi-\eta)^2\}[1-F(q)]^2, \qquad (2)$$

where $u=\epsilon_1\theta_1$, $v=\epsilon_2\theta_2$, ϵ_1 and ϵ_2 being the electron and positron energies, respectively, and θ_1 and θ_2 the angles between electron and photon and between positron and photon, respectively. $\xi=(1+u^2)^{-1}$, $\eta=(1+v^2)^{-1}$, $\mathbf{q}=\mathbf{k}-\mathbf{p}_1-\mathbf{p}_2$, $\mathbf{q}_\perp=-\mathbf{u}-\mathbf{v}$, and $F(q)$ is the atomic form factor. Energies are measured in units of mc^2, and momenta in units of mc.

In order to obtain the cross section as a function of the angle θ between the electron and the positron I should be integrated over angles Ω_1 with θ fixed. Expressed in terms of θ and θ_1, $q_\perp$ and v are given by

$$q_\perp^2=(k/\epsilon_1)^2\{u^2+w^2-2uw\cos\phi_1\} \qquad (3)$$

and

$$v^2=(k/\epsilon_1)^2\{(\epsilon_2/k)^2u^2+w^2-2(\epsilon_2/k)uw\cos\phi_1\}, \qquad (4)$$

where $w=(\epsilon_1\epsilon_2/k)\theta$ and ϕ_1 is the angle between the $\mathbf{p}_1-\mathbf{k}$ plane and the $\mathbf{p}_1-\mathbf{p}_2$ plane (see Fig. 1).

Replacing $d\Omega_2$ by $d\Omega=\theta d\theta d\phi$ we have the integral

$$\mathcal{J}(w)=\int Id\Omega_1 d\phi=\frac{2\pi}{\epsilon_1^2}\int d\phi_1 u du\frac{(1-F(q))^2}{q^4}\times\{(\epsilon_1^2+\epsilon_2^2)\xi\eta q_\perp^2+2\epsilon_1\epsilon_2(\xi-\eta)\}, \qquad (5)$$

and the cross section is given by

$$d\sigma(w,\epsilon_1)=(\sigma_0/2\pi^2)\mathcal{J}(w)(d\epsilon_1/k)wdw. \qquad (6)$$

3. THE CROSS SECTION $d\sigma(w,\epsilon)$ FOR NO SCREENING

The integral $\mathcal{J}(w)$ in Eq. (5) is most conveniently divided into two regions $q\sim 1$ and $q\sim 1/k$. The integrations are considerably more cumbersome than the integrations over Ω_1 with θ_2 fixed. For the case of no screening the result may be written

$$d\sigma(w,\epsilon_1)=\sigma_0\frac{d\epsilon_1}{k^3}wdw\zeta^2\times\left\{2\left(\ln\frac{2\epsilon_1\epsilon_2}{k}-\frac{\epsilon_1}{k}\ln\frac{\epsilon_1}{k}-\frac{\epsilon_2}{k}\ln\frac{\epsilon_2}{k}-\frac{\epsilon_1^2+\epsilon_2^2}{2k^2}\right)(\epsilon_1^2+\epsilon_2^2+4\epsilon_1\epsilon_2w^2\zeta^2)\right.$$
$$\left.+8\epsilon_1\epsilon_2w^2\zeta^2\left(\frac{\epsilon_1\epsilon_2(\epsilon_1-\epsilon_2)}{k^3}\ln\frac{\epsilon_2}{\epsilon_1}-\frac{\epsilon_1^2+\epsilon_2^2}{2k^2}-1\right)-\frac{\gamma}{\sinh\gamma}\left[(\epsilon_1^2+\epsilon_2^2)\zeta^{-1}+4\left(\frac{\epsilon_1\epsilon_2}{k}\right)^2\right]\right\}. \qquad (7)$$

We have here introduced $\zeta=(1+w^2)^{-1}$ analogously to the definitions of ξ and η, and γ is given by

$$\cosh(\gamma/2)=k(4\epsilon_1\epsilon_2\zeta)^{-1/2}. \qquad (7a)$$

In order to check the result Eq. (7) we integrate the cross section Eq. (7) over w, which gives

$$d\sigma(\epsilon_1)=\sigma_0\frac{d\epsilon_1}{k^3}(\epsilon_1^2+\epsilon_2^2+\tfrac{2}{3}\epsilon_1\epsilon_2)\left(\ln\frac{2\epsilon_1\epsilon_2}{k}-\frac{1}{2}\right),$$

which is the well-known energy distribution of the pair particles. Another check on Eq. (7) is obtained in Sec. 4.

4. BORSELLINO'S FORMULA

The formula given by Borsellino[12] may now be derived for the high-energy case by integrating the cross section Eq. (7) over the electron energy ϵ_1, keeping the invariant pair energy

$$4Q^2=(\epsilon_1+\epsilon_2)^2-(\mathbf{p}_1+\mathbf{p}_2)^2=k^2/(\epsilon_1\epsilon_2\zeta)$$

[13] See, for instance, Haakon Olsen and L. C. Maximon, Phys. Rev. **114**, 887 (1959) [note Eqs. (3.20), (4.5), and (4.10)].

fixed. In this way we find

$$d\sigma(Q)=\int_0^\infty dw\int_1^{k-1} d\epsilon_1\delta[w-(4Q^2\epsilon_1\epsilon_2/k^2-1)^{1/2}]\times\frac{d^2\sigma(w,\epsilon_1)}{dwd\epsilon_1}=4\sigma_0F\frac{dQ}{Q^3},\quad (8)$$

with

$$F=\ln(k/Q)\{L(2+2/Q^2-1/Q^4)-\Delta(1+1/Q^2)\}-(L/6)(16+21/Q^2-17/2Q^4)+(\Delta/12)(28+17/Q^2),$$

where $L=\cosh^{-1}Q$ and $\Delta=(1-1/Q^2)^{1/2}$. This is Borsellino's formula[12] for the high-energy case as corrected by Hart *et al.*[9] This provides an additional check on the correctness of Eq. (7).

It is now clear, however, that the cross section Eq. (8) is the distribution of Q and not of w. In other words, in order that Eq. (8) could be compared with measurements of w, experimental values of w and ϵ_1 should be combined in the quantity $Q=(k/2)(\epsilon_1\epsilon_2\zeta)^{-1/2}$ for all possible values of w and ϵ_1 giving the same value of Q. The resulting distribution of Q might then be compared with $d\sigma(Q)$ Eq. (8). Thus, Borsellino's formula cannot be compared with the measurements of references 1–11 in which θ and ϵ_1 are independent variables.

5. COMPARISON WITH THE DISTRIBUTION OF THE ANGLE BETWEEN PHOTON AND ELECTRON

It may now be shown that the w distribution Eq. (7) is almost equal to the u distribution, the distribution of the angle θ, between the electron (or the positron) and the photon. The latter is given by[14]

$$d\sigma(u,\epsilon_1)=\sigma_0\frac{d\epsilon_1}{k^3}udu\xi^2\left\{(\epsilon_1^2+\epsilon_2^2)\left(2\ln\frac{2\epsilon_1\epsilon_2}{k}-1\right)+2\epsilon_1\epsilon_2\left[1+4u^2\xi^2\left(\ln\frac{2\epsilon_1\epsilon_2}{k}-2\right)\right]\right\}.\quad (9)$$

The main term in Eqs. (7) and (9) is for large k the $\ln(2\epsilon_1\epsilon_2/k)$ term; neglecting for a moment all but this term, Eq. (7) becomes

$$d\sigma(w,\epsilon_1)=2\sigma_0\frac{d\epsilon_1}{k^3}wdw\zeta^2\ln(2\epsilon_1\epsilon_2/k)(\epsilon_1^2+\epsilon_2^2+4\epsilon_1\epsilon_2w^2\zeta^2),$$

and Eq. (9) reduces to

$$d\sigma(u,\epsilon_1)=2\sigma_0\frac{d\epsilon_1}{k^3}udu\xi^2\ln(2\epsilon_1\epsilon_2/k)(\epsilon_1^2+\epsilon_2^2+4\epsilon_1\epsilon_2u^2\xi^2).$$

The main terms in the w and u distributions are, thus, identical, and since most of the contribution to the cross section for high energies comes from the $\ln(2\epsilon_1\epsilon_2/k)$ term, we conclude that the two distributions are almost equal.

[14] Reference 13, Eqs. (10.4) and (6.23), (6.29), and (6.34).

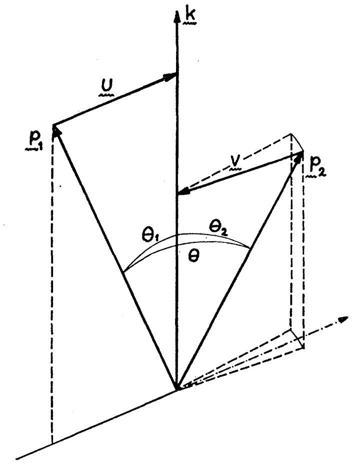

FIG. 2. The case of almost coplanarity of **k**, $\mathbf{p}_1$, and $\mathbf{p}_2$. $\theta_2=v/p_2\approx u/p_2$, and, therefore, $\theta\approx\theta_1+\theta_2\approx u(1/p_1+1/p_2)=uk/\epsilon_1\epsilon_2$; thus, $u\approx w$.

This fact that the main terms are identical may be understood by considering the integral Eq. (5). We show below that this is true also when screening is included. This allows us to deduce the effect of screening on $d\sigma(w,\epsilon_1)$ from the cross section $d\sigma(u,\epsilon_1)$ with screening included which is known from earlier work.[14]

The main term in the cross section, the term containing $\ln(2\epsilon_1\epsilon_2/k)$ for no screening comes from small values of the momentum transfer $q,q\sim 1/k$. Now the component of **q** along **k**, $q_z=\delta/\xi$ is alway s of the order $1/k$.[15] Here $\delta=k/(2\epsilon_1\epsilon_2)$. If also the component of **q** perpendicular to **k**, $q_\perp$, is to be small of order $1/k$, $\mathbf{p}_1$, $\mathbf{p}_2$, and **k** must be almost coplanar and u-v is of the order $1/k$.[15]

From Eq. (3) then follows that also v-w and ϕ_1 are of the order $1/k$ (see also, Fig. 2),

$$q_\perp^2=(k/\epsilon_1)^2[(v-w)^2+w^2\phi_1^2],\quad (10)$$

and from Eq. (4)

$$u^2-v^2=(k/\epsilon_1)(u^2-w^2).\quad (11)$$

Now the integral Eq. (5) is in the region $q\sim 1/k$, with $\gamma=(k/\epsilon_1)(u-w)$ and $\psi=(k/\epsilon_1)\phi_1$ found to be given by

$$\mathcal{G}(w)=\frac{2\pi w}{k^2}\int_{-\infty}^{\infty}d\psi\int_{-\infty}^{\infty}d\gamma\frac{[1-F(q)]^2}{q^4}\times\{(\epsilon_1^2+\epsilon_2^2)\zeta^2q_\perp^2+2\epsilon_1\epsilon_2(\xi-\eta)^2\},\quad (12)$$

where q and $\xi-\eta$ are from Eqs. (10) and (11)

$$q^2=\gamma^2+w^2\psi^2+(\delta/\zeta)^2,\quad (13)$$

$$\xi-\eta=2w\zeta^2\gamma.\quad (14)$$

[15] H. Olsen, L. C. Maximon, and H. Wergeland, Phys. Rev. **106**, 27 (1956).

On the other hand, the u distribution is obtained by integrating Eq. (1) over $d\Omega_2$ keeping u and ϵ_1 fixed

$$d\sigma(u,\epsilon_1)=(\sigma_0/2\pi^2)\mathcal{J}'(u)(d\epsilon_1/k)udu, \tag{15}$$

with

$$\mathcal{J}'(u)=\int Id\Omega_2d\phi_1=\frac{2\pi}{\epsilon_2^2}\int d\phi_2vdv\frac{[1-F(q)]^2}{q^4}\times\{(\epsilon_1^2+\epsilon_2^2)\xi\eta q_\perp^2+2\epsilon_1\epsilon_2(\xi-\eta)^2\}. \tag{16}$$

In the region $q\sim 1/k$ $\mathcal{J}'(u)$ reduces to, with $\nu=u-v$,

$$\mathcal{J}'(u)=\frac{2\pi u}{\epsilon_2^2}\int d\phi_2d\nu\frac{[1-F(q)]^2}{q^4}\times\{(\epsilon_1^2+\epsilon_2^2)\xi^2q_\perp^2+2\epsilon_1\epsilon_2(\xi-\eta)^2\},$$

where now q and $\xi-\eta$ are given by

$$q^2=\nu^2+u^2\phi_2^2+(\delta/\xi)^2 \tag{17}$$

and

$$\xi-\eta=2u\xi^2\nu. \tag{18}$$

Comparing now Eqs. (12), (13), and (14) to (16), (17), and (18) we see that the integrand of Eq. (12) is identical to that of Eq. (16), when in the former u is substituted for w. Therefore, $\mathcal{J}(w)$ is obtained from $\mathcal{J}'(u)$ by substituting in $\mathcal{J}'(u)$ w for u and multiplying with $(\epsilon_2/k)^2$. We, thus, find from Eq. (15) the contribution to the cross section $d\sigma(w,\epsilon_1)$ from the region $q\sim 1/k$

$$d\sigma(w,\,\epsilon_1,\,q\sim 1/k)=(\sigma_0/2\pi^2)\mathcal{J}'(w)(d\epsilon_1/k)wdw,$$

which shows that the distributions of w and u are identical in the region $q\sim 1/k$.

It should also be noted that the main term in the distribution of the angle θ is obtained from the θ_1 distribution by multiplying the argument $\epsilon_1\theta$ of $\mathcal{J}'(\epsilon_1\theta)$ by ϵ_2/k. In particular then for the case of even energy partition $\epsilon_1=\epsilon_2=k/2$, the distribution $d\sigma(\theta,\epsilon_1)$ is twice as wide as the $d\sigma(\theta_1,\epsilon_1)$ distribution. A conjecture of Bradt *et al.*[16] is, thereby, proved. At extremely high energies they equated the mean value of the angle between the electron and positron to twice the mean value of the angle between the electron and the photon.

That the cross section $d\sigma(u,\epsilon_1)$ and $d\sigma(w,\epsilon_1)$ have almost the same angular dependence also for finite values of k is shown in Fig. 3, where the two cross sections are plotted as functions of u and w for a photon energy $k(mc^2)=50$ MeV.

6. THE CROSS SECTION $d\sigma(w,\epsilon_1)$ WITH SCREENING INCLUDED

As the result of the previous section is true for any amount of screening, and since screening only affects the cross sections for $q\sim 1/k$, it follows that the effect of screening on $d\sigma(w,\epsilon_1)$ is the same as on $d\sigma(u,\epsilon_1)$. Thus, screening is included in $d\sigma(w,\epsilon_1)$ by substituting for $\ln(2\epsilon_1\epsilon_2/k)$ in Eq. (7)

$$\mu=\ln(2\epsilon_1\epsilon_2/k)+\mathfrak{F}(\delta/\zeta)\quad\text{(part.scr.)} \tag{19}$$

for partial screening, where $\mathfrak{F}(\delta/\zeta)$ is given earlier.[17] In particular for complete screening[18]

$$\mu=\ln(111Z^{-1/3}/\zeta).\quad\text{(compl.scr.)} \tag{20}$$

In general then, from Eq. (7)

$$d\sigma(w,\epsilon_1)=\sigma_0\frac{d\epsilon_1}{k^3}wdw\zeta^2\times\Bigg\{2\left(\mu-\frac{\epsilon_1}{k}\ln\frac{\epsilon_1}{k}-\frac{\epsilon_2}{k}\ln\frac{\epsilon_2}{k}-\frac{\epsilon_1^2+\epsilon_2^2}{2k^2}\right)[\epsilon_1^2+\epsilon_2^2+4\epsilon_1\epsilon_2\zeta(1-\zeta)]+8\epsilon_1\epsilon_2\zeta(1-\zeta)\left[\frac{\epsilon_1\epsilon_2(\epsilon_1-\epsilon_2)}{k^3}\ln\frac{\epsilon_2}{\epsilon_1}-\frac{\epsilon_1^2+\epsilon_2^2}{2k^2}-1\right]-\frac{\gamma}{\sinh\gamma}\left[(\epsilon_1^2+\epsilon_2^2)\zeta^{-1}+4\left(\frac{\epsilon_1\epsilon_2}{k}\right)^2\right]\Bigg\}, \tag{21}$$

with μ given by Eqs. (19) and (20) for partial and complete screening, respectively. γ is defined in Eq. (7a).

7. EQUIPARTITION OF ENERGY

For comparison with experiments the case of equal energy sharing between the electron and the positron, $\epsilon_1=\epsilon_2=k/2$, is of special interest. In this case the cross section Eq. (21) becomes

$$d\sigma\left(w,\epsilon_1=\frac{k}{2}\right)=\sigma_0\frac{d\epsilon_1}{k^3}wdw\zeta^2\Bigg\{(\mu+\ln 2)[1+2\zeta(1-\zeta)]-[\tfrac{1}{4}+3\zeta(1-\zeta)]-\frac{\gamma}{4\sinh\gamma}(3+2w^2)\Bigg\}, \tag{22}$$

where $\cosh\gamma/2=\zeta^{-1/2}$, and

$$\mu=\begin{cases}\ln(k/2), & \text{(no screening)},\\ \ln(k/2)+\mathfrak{F}(2/k\zeta), & \text{(partial screening)}\\ \ln(111Z^{-1/3}/\zeta) & \text{(complete screening)}.\end{cases} \tag{23}$$

The form of the cross section as a function of w, the w distribution, does not change much with the energy of the photon as shown in Fig. 4 for the case of equipartition of energy.

In Figs. 5 and 6 the w distribution Eq. (22) is compared with experiments. From the work of Schopper[11] it is seen that even at a photon energy as low as $k=6$

[16] H. L. Bradt, M. F. Kaplon, and B. Peters, Helv. Phys. Acta **23**, 24 (1950).
[17] Reference 13, Eqs. (6.31) and Table I.
[18] Reference 13, Eq. (6.34).

MeV the agreement between theory and experiment is very good.

Finally, in Fig. 7 it is shown that the form of the w distribution does not change much with energy partition. Therefore, the curves of Fig. 4 may be used for comparison with experiments in which pairs of energy partitions different from $\epsilon_1=\epsilon_2$ are included. It may, in fact, be shown that the cross section Eq. (21) integrated over all energies ϵ_1 with w kept fixed,

$$d\sigma(w)=\int_1^{k-1}\left[\frac{d\sigma(w,\epsilon_1)}{d\epsilon_1}\right]d\epsilon_1,$$

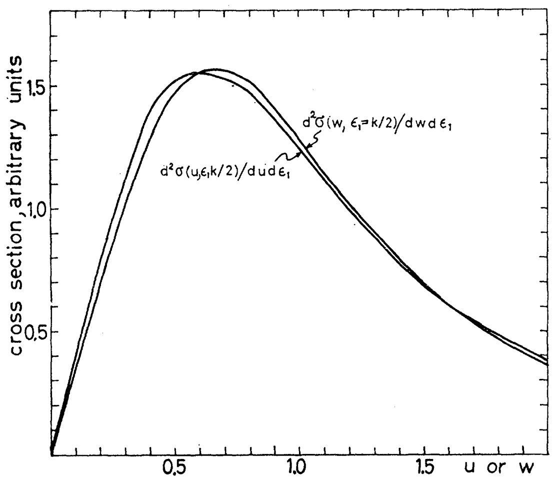

FIG. 3. $d^2\sigma(w,\epsilon_1)/dwd\epsilon_1$ and $d^2\sigma(u,\epsilon_1)/dud\epsilon_1$ for no screening for $k=50$ MeV and equipartition of energy $\epsilon_1=\epsilon_2=k/2$. The curves have been normalized to the same value at $u=w=0.6$.

differs only little from the w distributions for equipartition of energy Fig. 4.

It should on the other hand be noted that although the form of the cross section does not depend strongly on w, the dependence on $Q=(k/2)(\epsilon_1\epsilon_2\zeta)^{-1/2}$ is very pronounced. In fact, the more the energy partition departs from equipartition, the more $d\sigma(Q,\epsilon_1)$ is shifted towards larger values of Q, which explains why the Borsellino distribution Eq. (8) is always broader than the experimental distribution $d\sigma(w,\epsilon_1)$.

8. THE MEAN OPENING ANGLE

The mean opening angle between the electron and the positron defined by

$$\langle w\rangle=\int_0^\infty wdw\left[\frac{d\sigma(w,\epsilon_1)}{dw}\right]\Big/\int_0^\infty dw\left[\frac{\sigma(w,\epsilon_1)}{dw}\right] \quad (24)$$

is found from Eq. (22) for equipartition of energy to be given by

$$\langle w\rangle=\frac{\pi}{2}\frac{15}{16}\frac{\ln k-41/30}{\ln(k/2)-\frac{1}{2}}, \quad (25)$$

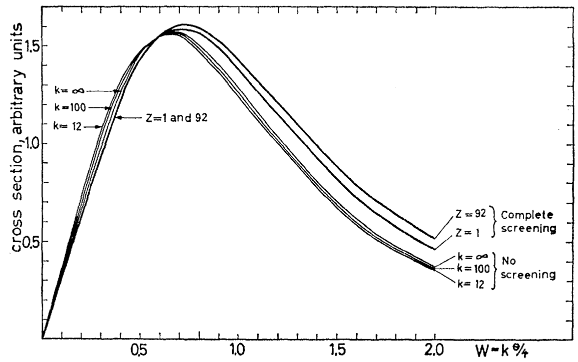

FIG. 4. Variation of w distribution with photon energy for $\epsilon_1=\epsilon_2=k/2$. Photon energies given are in units of mc^2. The curves have been normalized to the same value at $w\equiv k\theta/4=0.6$.

for no screening, and

$$\langle w\rangle=\frac{\pi}{2}\frac{15}{16}\frac{\ln(888Z^{-1/3})-\frac{3}{5}}{\ln(182Z^{-1/3})-1/24}, \quad (26)$$

for complete screening.

In Fig. 8 curves for $\langle w\rangle$ are shown for $Z=1$, 8, and 90. The values for $\langle w\rangle$ for intermediate screening have been obtained using values of $\mathfrak{F}(\delta/\zeta)$ given before.[17] In Fig. 8 also the available measured values of $\langle w\rangle$ at high energies have been plotted. The agreement with theory is reasonable above $k\approx 25$ MeV, when it is considered that multiple scattering and experimental uncertainties always will tend to increase the experimental value of $\langle w\rangle$.

This is indicated for the measurements of Hart *et al.*[9] where the multiple scattering theory of Appendix B has been used to correct for multiple scattering and experimental resolution (points marked "theory" in Fig. 8). The experimental resolution has been assumed to be Gaussian with widths as given in the paper of Hart *et al.*[9]

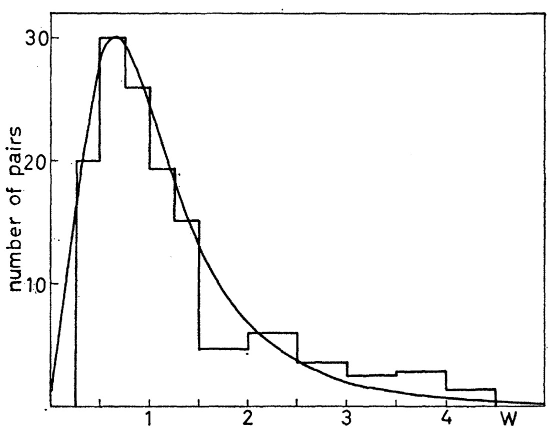

FIG. 5. Theoretical w distribution, Eq. (22), for $k=50$ MeV compared to experimental w distribution of Hintermann (reference 7).

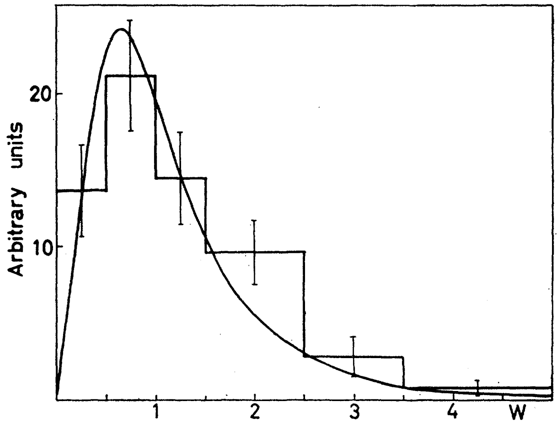

FIG. 6. Theoretical w distribution, Eq. (22), for $k=100$ MeV compared to experimental w distribution of Baroni *et al.* (reference 6).

Below $k\approx 25$ MeV the theory gives a too high value on $\langle w\rangle$, in spite of the fact that even for $k=6$ MeV the agreement with the distribution of w is very good.[11] The reason for this is that in $\langle w\rangle$ relatively large angles are important. For these angles, however, the high-energy approximation applied here becomes poor.

In Appendix A the large angle contribution to $\langle\theta\rangle$ is calculated approximately. The correction is only important for so low values of k that screening is unimportant. The mean value $\langle w\rangle$ Eq. (25) is multiplied with the large angle correction factor $(1-24/5k)$,

$$\langle w\rangle=\frac{\pi}{2}\frac{15}{16}\frac{\ln k-41/30}{\ln(k/2)-\frac{1}{2}}\left(1-\frac{24}{5k}\right). \qquad (27)$$

This correction factor is taken into account in curve (b) of Fig. 8. The agreement with experiments is now good.

9. PHOTON ENERGY ESTIMATE BY OPENING ANGLE MEASUREMENT

The experimental method of using the opening angle of electron positron pairs to estimate the energy of the photon creating the pair was introduced by Bradt *et al.*[16] The good agreement between experiment and theory of Secs. 7 and 8 gives one new confidence in this method. In order to apply the theory of the preceding sections to this method, we must first integrate the cross section over all energies ϵ_1 with θ fixed and secondly, the effect of multiple scattering must be taken into account.

The result of the integration over ϵ_1 is shown in Fig. 9. The w_0 distribution of the integrated cross section $d\sigma(w_0)/dw_0$, with $w_0=k\theta/4$, is seen to be very close to the distribution for equipartition of energy $d^2\sigma(w,\epsilon_1=k/2)/dwd\epsilon_1$.

It is clear that the w_0 distribution, $d\sigma(w_0)/dw_0$, when normalized, represents the probability distribution, for a given value of θ, that k has a value $k=2w_0/\theta$. The most probable value of w_0 is according to Fig. 9 $w_{0,\max}=0.8$.

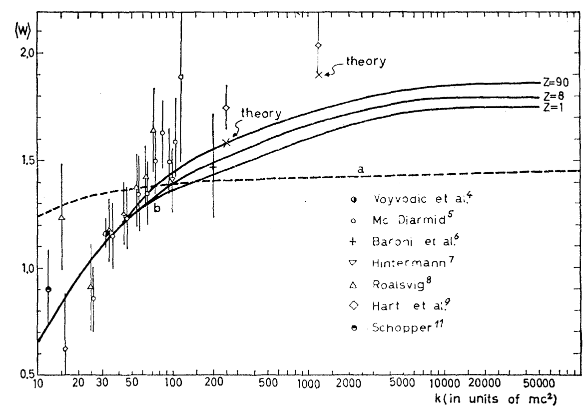

FIG. 8. $\langle w\rangle$ as a functon of k for $\epsilon_1=\epsilon_2=k/2$. Curve a is calculated for no screening, Eq. (25). In curve b are included, for energies above about $k=40$, corrections for screening and for energies below about $k=100$, corrections for large-angle contributions [Eq. (27)]. In the experiment of Hart *et al.* (reference 9) the pairs were created in hydrogen vapor; in all the other experiments, the pairs were created in photographic emulsion. Points marked "theory" are calculated values of $\langle w\rangle$ for the experiment of Hart *et al.* (reference 9) including multiple scattering and experimental resolution.

The most probable value of k for a measured opening angle θ is thus

$$k=3.2/\theta,$$

or, since k is measured in units of mc^2,

$$k_{\mathrm{MeV}}=1.6/\theta. \qquad (28)$$

Multiple scattering will change this value of k. According to Appendix B, the quantity determining the effect of multiple scattering is

$$a^2=(\tfrac{1}{3})\langle\theta_s^2\rangle(k/4)^2L. \qquad (29)$$

When $a\ll 1$ multiple scattering gives only a small correction to the most probable value of k, Eq. (28). On the other hand, if $a\gg 1$, then the pair opening angle is mainly determined by the multiple scattering of the electron and positron.

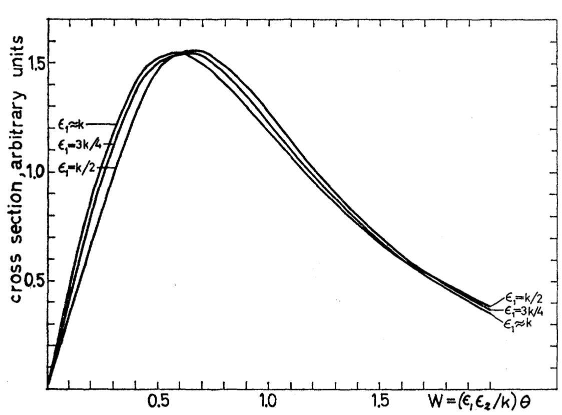

FIG. 7. Variation of w distribution with energy partition for the case of no screening for a photon energy $k=50$ MeV. The curves have been normalized to the same value at $w=0.6$. For the case $\epsilon_1\approx k$, it is assumed that $\epsilon_2\ll k$, but still $\epsilon_2\gg 1$.

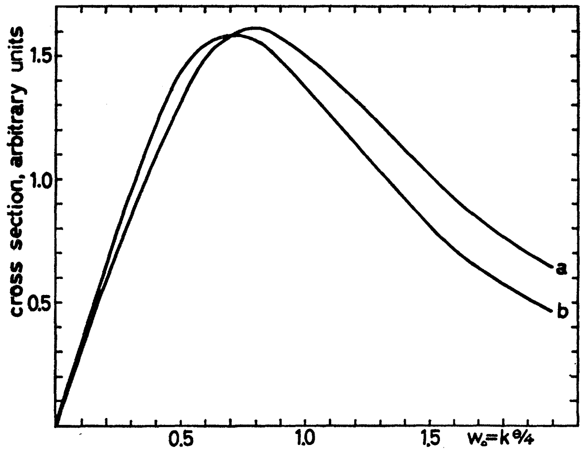

FIG. 9. Cross section $d^2\sigma(\theta,\epsilon_1)/d\theta d\epsilon_1$ numerically integrated over energies ϵ_1 as a function of $w_0=k\theta/4$ (curve a) for complete screening. For comparison, the cross section $d^2\sigma(\theta,\epsilon_1)d\theta d\epsilon_1$ for equipartition of energy $\epsilon_1=\epsilon_2=k/2$ (complete screening) is also shown (curve b). The curves have been calculated for $Z=1$, but, for the purpose of Sec. 9, they might be used for any element with negligible error. The curves have been normalized to the same value at $w_0=0.7$.

$\langle\theta_s^2\rangle$ may be found from curves[19] computed from Moliere's theory.[20] Less accurate but simpler formula for a^2 might be found using the formula for $\langle\theta_s^2\rangle$ given in Rossi's book[21]

$$a^2=0.21(Z^2/A)(\ln\gamma)L_{(\mathrm{g/cm^2})}, \tag{30}$$

where γ is the smaller of the two numbers, $196(ZA)^{-1/6}$ and $[268Z^{-1/3}k]^{1/2}$, Z and A are the atomic number and atomic weight of the scatterer, respectively. L in Eq. (30) is given in g/mc^2. For pairs created in photographic emulsion one finds, using a scattering constant of 26,[22]

$$a^2=2.2\ 10^{-3}L_{(\mu)}, \tag{31}$$

where $L_{(\mu)}$ is given in microns.

Using the formula (B5) of Appendix B, we find for not too large values of L, the most probable value of k for a measured opening angle θ,

$$k_{\mathrm{MeV}}=1.6(1+\tfrac{5}{2}a^2)/\theta. \tag{32}$$

Correspondingly from Eq. (B7) we get for large values of L,

$$k_{\mathrm{MeV}}=2a[1+(\ln 2a^2+1.29)/4a^2]/\theta. \tag{33}$$

It should be noted that for pairs created in photographic emulsion this gives using Eq. (31)

$$k_{\mathrm{MeV}}=0.094(L_{(\mu)}{}^{3/2}/d_{(\mu)}) \times\{1+[\ln(L_{(\mu)}10^{-3})+2.77]/(8.8\times10^{-3}L_{(\mu)})\}, \tag{34}$$

where $d_{(\mu)}$ (measured in μ) is the distance between the electron and positron at a distance $L_{(\mu)}$ from the origin of the pairs. For large values of $L_{(\mu)}$, Eq. (33) gives a considerably smaller value of k_{MeV}, than the formula in common use[23] $k_{\mathrm{MeV}}=L_{(\mu)}{}^{3/2}/d_{(\mu)}$.

An interpolation formula for values of L for which Eqs. (32) and (33) fail to be valid is given by

$$k_{\mathrm{MeV}}=2(0.62+a^2)^{1/2}(1+\Delta)/\theta, \tag{35}$$

where

$$\Delta=0.017\frac{1+19.4a^2+62.7a^4}{1+4.27a^2+7.9a^4+6.12a^6}.$$

It might be shown that the simple formula

$$k_{\mathrm{MeV}}=2(L_{(\mu)}/d_{(\mu)})(0.62+a^2)^{1/2} \tag{36}$$

gives the most probable value of k_{MeV} for all values of L with an error which is less than 10%.

Finally, we consider briefly the degree of confidence which might be given to the values of k obtained from Eqs. (32)–(36).

For not too large values of L, the w_0 distribution might for the present purpose be approximated by, as discussed in Appendix B,

$$P(w_0)=2w_0(1+w_0^2)^{-2},$$

and the value found from Eq. (32) is with the uncertainty $\pm\Delta k$ given with the confidence

$$C(\Delta k)=\int_{(k-\Delta k)\theta/4}^{(k+\Delta k)\theta/4}P(w_0)dw_0=\{[1+0.64(1-\Delta k/k)^2]^{-1} -[1+0.64(1+\Delta k/k)^2]^{-1}\}. \tag{37}$$

For $\Delta k/k=\frac{1}{2}$ the confidence is 45% and for $\Delta k/k=\frac{3}{4}$, 62%.

Correspondingly, for large values of L the w_0 distribution is approximately

$$P(w_0)=a^{-2}w_0e^{-w_0^2/2a^2},$$

and the confidence is

$$C(\Delta k)=\exp[-\tfrac{1}{2}(1-\Delta k/k)^2] -\exp[-\tfrac{1}{2}(1+\Delta k/k)^2]. \tag{38}$$

For $\Delta k/k=\frac{1}{2}$ the confidence is here 45% and for $\Delta k/k=\frac{3}{4}$, 75%. From the experiment of Hintermann[7] we find for 135 measured pairs of photon energies between 10 and 1000 MeV for $\Delta k/k=\frac{1}{2}$, a confidence of 55% and for $\Delta k/k=\frac{3}{4}$, a confidence of 70%, which are close to the values of $C(\Delta k/k)$ from Eqs. (37) and (38).

ACKNOWLEDGMENTS

The author is indebted to Dr. Herwig Schopper for a remark leading to the re-examination of the opening angle formula, and for communication of experimental results prior to publication. Conversations with T. Jacobsen of the University of Oslo concerning the experimental method of determining photon energies from opening angle measurements are gratefully acknowledged.

[19] Helge Øverås, CERN Report 60–18, 1960 (unpublished).
[20] G. Molière, Z. Naturforsch. **3a**, 78 (1948).
[21] B. Rossi, *High-Energy Particles* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1952) [note Eqs. (4) and (5), p. 67].
[22] See, for example, K. Gottstein, in *Kosmische Strahlung*, edited by W. Heisenberg (Springer-Verlag, Berlin, 1953), p. 515.
[23] C. E. Powell, P. H. Fowler, and D. H. Perkins, in *The Study of Elementary Particles by the Photographic Method* (Pergamon Press, Inc., New York, 1959), p. 192.

APPENDIX A. PAIR PRODUCTION AT LARGE ANGLES

(A) The Angle between Photon and Electron

It is easy to see that the error in the cross section $d\sigma(u,\epsilon_1)$ Eq. (9) at large angles and high energies is of the order $1/k^2$. Thus, the correction to the average angle $\langle\theta\rangle$ from large angles will be of the order $1/k$.

Indeed, from the work of Hough[24] we have the large-angle high-energy cross section for an angle θ_1 between the photon and the electron at a given energy partition

$$d\sigma(\theta_1,\epsilon_1)=\tfrac{1}{8}\sigma_0\frac{d\epsilon_1}{k^3}\sin\theta_1 d\theta_1\Big\{(1+\gamma^2)[\cot^2(\tfrac{1}{2}\theta_1)/\sin^2(\tfrac{1}{2}\theta_1)]\ln(2\epsilon_1\epsilon_2/k)+(\gamma/4)(5-2\gamma)/\sin^4(\tfrac{1}{2}\theta_1)$$
$$-(\gamma^2/4\rho^2)(2+\gamma)/\sin^4(\tfrac{1}{2}\theta_1)+[\gamma^2-2\ln(1+\gamma)]/\sin^2(\tfrac{1}{2}\theta_1)$$
$$+[(1+\gamma)/2\rho^3][3\rho^3-\gamma(2+\gamma)]\ln[(\rho+\gamma)/(\rho-\gamma)]/\sin^2(\tfrac{1}{2}\theta_1)\Big\},\quad \text{(A1)}$$

with $\gamma=\epsilon_2/\epsilon_1$ and $\rho^2=\gamma^2+4(1+\gamma)\sin^2(\frac{1}{2}\theta_1)$. It is easy to see that Eq. (A1) as $\theta_1\to 0$, and Eq. (9) as $u=\epsilon_1\theta_1\to\infty$, approach the same limit

$$d\sigma(\theta_1\epsilon_1)=\sigma_0\frac{d\epsilon_1}{k^3}\left(\frac{du}{u^3}\right)\left\{2(\epsilon_1^2+\epsilon_2^2)\ln\left(\frac{2\epsilon_1\epsilon_2}{k}\right)-(\epsilon_1-\epsilon_2)^2\right\}.$$

The two cross sections may, thus, be used for calculating the average $\langle\theta_1\rangle$ for all angles. The result is for $\epsilon_1=\epsilon_2=k/2$

$$\langle u\rangle=\frac{\pi}{2}\frac{15}{16}\frac{\ln(k/2)-\frac{2}{5}}{\ln(k/2)-\frac{1}{2}}\left(1-\frac{12}{5k}\right),\quad \text{(A2)}$$

where the last factor is the correction from large angles.

(B) The Angle between Electron and Positron

The integration of the Bethe-Heitler formula with the fixed angle θ between the electron and the positron large, is considerably more complicated than the corresponding case under (a) above. We only calculate the correction to the term ln k in $\langle\theta\rangle$ and assume that the correction to the constant term $-41/30$ in Eq. (25) is the same.

By inspection of the differential Bethe-Heitler cross section it is seen that the term containing ln k in Eq. (A1)

$$d\sigma_1(\theta_1\epsilon_1)=\tfrac{1}{8}\sigma_0\frac{d\epsilon_1}{k^3}\sin\theta_1 d\theta_1(1+\gamma^2)\times[\cot^2(\tfrac{1}{2}\theta_1)/\sin^2(\tfrac{1}{2}\theta_1)]\ln k,\quad \text{(A3)}$$

like in the case of small angles θ_1 comes from integration over θ_2, the positron angle for which the electron, positron, and photon are almost coplanar. Moreover, since the angle θ_1 is assumed to be large, the only case of coplanarity giving contributions to the cross section of order $1/k^2$ is when $\mathbf{p}_2$ is almost parallel to $\mathbf{k}$. (See Fig. 10.)

Turning now to the cross section for an angle θ between the electron and the positron $d\sigma(\theta_1\epsilon_1)$, the θ distribution of the ln k term must be the same as that of Eq. (A1), since when $\mathbf{p}_2$ is almost parallel to $\mathbf{k}$, θ and θ_1 are almost equal. Since, however, with θ fixed either $\mathbf{p}_1$ or $\mathbf{p}_2$ may be parallel to $\mathbf{k}$ we must add to Eq. (A3) a term with ϵ_1 and ϵ_2 interchanged. There is also a third term, $f(\theta_1\epsilon_1)$, which secures that the small angle limit coincides with the large w limit of Eq. (7). $f(\theta,\epsilon_1)$ is, however, very small at large angles. We find

$$d\sigma_1(\theta,\epsilon_1)=\tfrac{1}{8}\sigma_0\frac{d\epsilon_1}{k^3}\sin\theta_1 d\theta_1\ln k\times\left\{[(1+\gamma^2)+(1+\gamma^{-2})]\frac{\cot^2\theta/2}{\sin^2\theta/2}+f(\theta,\epsilon_1)\right\},$$

where

$$f(\theta,\epsilon_1)=\left(\frac{\epsilon_1}{\epsilon_2}+\frac{\epsilon_2}{\epsilon_1}\right)\frac{32}{\theta^2}\quad\text{for }\theta\text{ small.}$$

and

$$f(\theta,\epsilon_1)\sim 1/k^2\quad\text{for }\theta\text{ large.}$$

It may easily be shown that for small angles Eq. (A4) approaches the same value as the ln k term of Eq. (7) approaches for large w.

For $\epsilon_1=\epsilon_2=k/2$, $d\sigma_1(\theta,\epsilon_1)$ is for large angles exactly equal to twice the ln k term $d\sigma(\theta_1\epsilon_1)$ Eq. (A3). Thus, the correction to the ln k term of $\langle\theta\rangle$ is exactly twice that of the correction to ln k term of $\langle\theta\rangle$, that is,

$$(1-24/5k).$$

Assuming that the correction to the term $-41/30$ is the same we obtain

$$\langle w\rangle=\frac{\pi}{2}\frac{15}{16}\frac{\ln k-41/30}{\ln(k/2)-\frac{1}{2}}\left(1-\frac{24}{5k}\right).\quad \text{(A5)}$$

APPENDIX B. EFFECT OF MULTIPLE SCATTERING

As a result of multiple scattering of the electron and positron the apparent opening angle will be increased.[7] The apparent opening angle is given by

$$\theta=d/L,$$

where d is the distance between the tracks at a distance L from the point of production of the pair.

From Fig. 11 it is seen that when multiple scattering

[24] P. V. C. Hough, Phys. Rev. 74, 80 (1948).

is taken into account the distribution of angles is given by

$$\frac{d^2\sigma_m(w)}{d\Omega d\epsilon_1}=\int\frac{d^2\sigma(w')}{dw'd\epsilon_1}P(y,a)\times P(|\mathbf{y}+(\mathbf{w}-\mathbf{w}')L|,b)w'dw'd\phi'ydyd\phi_y, \quad \text{(B1)}$$

where[25]

$$P(y,a)=\pi^{-1}a^{-2}e^{-y^2/a^2} \quad \text{(B2)}$$

is the probability for a deviation y from the undisturbed electron (or positron) path, and $d^2\sigma/dwd\epsilon_1$ is given by Eq. (21).

Quantities which have been corrected for multiple scattering are denoted by an index m. In what follows we take the case of equal energy $\epsilon_1=\epsilon_2=k/2$. Then the width of the distribution in the same units as w is given by[26]

$$a^2=b^2=(\tfrac{1}{3})\langle\theta_s^2\rangle(k/4)^2L, \quad \text{(B3)}$$

where $\langle\theta_s^2\rangle$ is the mean square spatial single scattering angle.

Since $d\sigma/d\Omega$ as a function of w does not differ much from the function const $\times\zeta^2$, we shall in this section for simplicity replace $d^2\sigma/d\Omega d\epsilon_1$ by const $\times\zeta^2$. The multiple scattering correction thus obtained, will then represent the multiple scattering correction to $d\sigma/d\Omega$ in a fairly good approximation. A check on this approximation is given below. Note that since the distribution of angles after integration over ϵ_1 (Sec. 9) differs little from the distribution for equal energies, the multiple scattering corrections obtained here will be valid also for the cross section integrated over ϵ_1, $d\sigma(w_0)/dw_0$ of Sec. 9.

In this way we find from Eq. (31), performing the integration over $\mathbf{y}$ and introducing the new variable $\mathbf{W}=\sqrt{2}a(\mathbf{w}'-\mathbf{w})$, the multiple scattering correction factor

$$\begin{aligned}f(w)&=(d^2\sigma_m/d\Omega d\epsilon_1)/(d^2\sigma/d\Omega d\epsilon_1)\\&=(2\pi a^2\zeta^2)^{-1}\int_0^{2\pi}d\phi'\int_0^\infty w'dw'(1+w'^2)^{-2}\\&\quad\times\exp\left[-\frac{(\mathbf{w}-\mathbf{w}')^2}{2a^2}\right]=\pi^{-1}\zeta^{-2}\int_0^{2\pi}d\phi'\int_0^\infty WdWe^{-W^2}\\&\quad\times[1+w^2+2a^2W^2+2\sqrt{2}a\mathbf{W}\cdot\mathbf{w}]^{-2}.\end{aligned}$$

For small values of a^2 we obtain, expanding in terms of a^2

$$f=1+4a^2\zeta(2-3\zeta)+24a^2\zeta^2(3-12\zeta+10\zeta^2)+O(a^6). \quad \text{(B4)}$$

As a result of this correction the value of w for which the cross section is a maximum will have the correction

[25] Reference 21, Eq. (8), p. 71.

[26] Reference 21, Eq. (8), p. 71 and Eq. (12), p. 72.

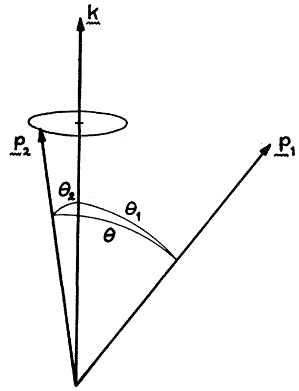

FIG. 10. Case of almost coplanarity for large values of θ_1. $\mathbf{p}_2$ is almost parallel to $\mathbf{k}$ and $\theta\approx\theta_1$.

factor, keeping only the lowest order of a^2,

$$w_{\max,m}/w_{\max}\approx 1+\tfrac{5}{2}a^2+O(a^4). \quad \text{(B5)}$$

If we had kept not only ζ^2 but the whole expression $\zeta^2+2\zeta^3(1-\zeta)$ multiplying $\ln 2k$ we would have obtained instead of Eq. (B5)

$$w_{\max,m}/w_{\max}=1+2.2a^2,+O(a^4),$$

which shows that our approximation is sufficiently good. In the same way it might be shown that other terms which we have neglected, for instance the term $\ln(1+w^2)$ in the case of complete screening, will have negligible effects on the multiple scattering corrections factors.

Correspondingly, we find for the case that multiple scattering is large ($a\gg 1$)

$$f(w)=(2\pi a^2\zeta^2)^{-1}e^{-w^2/2a^2}\times\left\{1+\frac{1}{2a^2}\left[g(w)-e^{w^2/2a^2}+1+\frac{w^2}{2a^2}\right]+O(1/4a^4)\right\}. \quad \text{(B6)}$$

Here

$$g(w)=Ei(-1/2a^2)+Ei(w/2a^2)-C-\ln(w/2a^2),$$

where $Ei(z)$ is the exponential integral,[27] and $C=0.577$.

The value of w for which the distribution is a maximum is for large values of a

$$w_{\max}=a\{1+[\ln(2a^2)+1.29]/4a^2+O(1/4a^4)\}. \quad \text{(B7)}$$

For the mean value $\langle w_m\rangle$ we find for small values of a^2 from Eq. (B4)

$$\langle w_m\rangle/\langle w\rangle=1+(\tfrac{1}{2})a^2-(\tfrac{3}{8})a^4+O(a^6). \quad \text{(B8)}$$

For the mean value it is possible to find an exact expression for any value of a^2. To this end we note that since the distribution function $d^2\sigma_m/d\Omega dk$ depends only on the magnitude of w, the mean value of the projected angle $|w_x|$ differs from the mean value of w only by a

[27] W. Magnus and F. Oberhettinger, *Formeln und Sätze für die speziellen Funktionen der mathematischen Physik* (Springer-Verlag, Berlin, 1948), 2nd ed. (note p. 127).

constant factor

$$\langle w\rangle=\frac{\pi}{2}\langle|w_x|\rangle.$$

The multiple scattering correction to $\langle w\rangle$ is, therefore, the same as to $\langle|w_x|\rangle$.

The correction to the projected distribution $d^2\sigma_m(w_x)/dw_x dk$ is from Eq. (B1), writing $w'dw'd\phi'=dw_x'dw_y'$, multiplying by dw_y and integrating over w_y and w_y':

$$\begin{aligned}f(w_x)&=[d^2\sigma_m(w_x)/dw_xdk]/[d^2\sigma(w_x)/dw_xdk]\\&=(2a^2\pi)^{-1/2}(1+w_x^2)^{3/2}\\&\quad\times\int_{-\infty}^{\infty}dw_x'(1+w_x'^2)e^{-(w_x-w_x')^2/2a^2}.\end{aligned}$$

From this the correction to the mean value of $|w_x|$ is

$$\begin{aligned}&\langle|w_x|_m\rangle/\langle|w_x|\rangle\\&=(2a^2\pi)^{-1/2}\int_0^{\infty}w_xdw_x\int_{-\infty}^{\infty}dw_x'(1+w_x'^2)^{-3/2}\\&\qquad\times\exp[-(w_x-w_x')^2/2a^2]\\&=(2a^2\pi)^{-1/2}\int_{-\infty}^{\infty}dW\exp-W^2/2a^2\\&\qquad\times\int_W^{\infty}dx(x-W)(1+x^2)^{-3/2},\end{aligned}$$

where we have introduced the new variables W and x by $w_x'=w_x+W$ and $w_x=x-W$. The x integration is elementary and gives

$$\begin{aligned}\langle|w_x|_m\rangle/\langle|w_x|\rangle&=(2a^2\pi)^{-1/2}\int_{-\infty}^{\infty}dWe^{-W^2/2a^2}(1+W^2)^{1/2}\\&=(2a^2\pi)^{-1/2}\int_0^{\infty}dt\,t^{-1/2}(1+t)^{1/2}e^{-t/2a^2},\end{aligned}$$

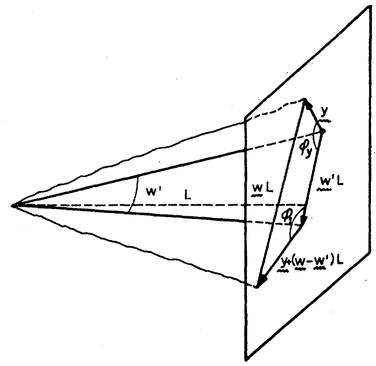

FIG. 11. Effect of multiple scattering. Opening angle for undisturbed electron and positron paths is w' (with angles measured in units of $\epsilon_1\epsilon_2/k$). At a distance L from the origin of the pair the distance between these paths is $w'L$. Multiple scattering causes deviations from the undisturbed paths of magnitudes $\mathbf{y}$ and $\mathbf{y}+(\mathbf{w}-\mathbf{w}')L$. The distance between the electron and positron is thereby changed to wL.

or

$$\langle w_m\rangle/\langle w\rangle=(2a^2)^{1/2}e^{1/4a^2}W_{1/2,1/2}(1/2a^2),\qquad\text{(B9)}$$

where $W_{1/2,1/2}(z)$ is the Whittaker function.[28]

Using the properties of[29] $W_{1/2,1/2}(z)$, we find for small a^2 the asymptotic expansion

$$\langle w_m\rangle/\langle w\rangle=1+\tfrac{1}{2}a^2-\tfrac{3}{8}a^4+\tfrac{5}{16}a^6-\cdots,\qquad\text{(B10)}$$

of which the first three terms coincide with Eq. (B8). For not too small values of a^2 the following formula is useful[29]:

$$\begin{aligned}\langle w_m\rangle/\langle w\rangle=(2a^2/\pi)^{1/2}\Bigg\{1+(1/2\sqrt{\pi})\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{n!(n+1)!}\\\times(2a^2)^{-n-1}[\psi(n+1)+\psi(n+2)\\-\psi(n+\tfrac{1}{2})+\ln(2a^2)]\Bigg\},\qquad\text{(B11)}\end{aligned}$$

where $\psi(z)=d[\ln\Gamma(z)]/dz$.

[28] Reference 28, p. 115.
[29] Reference 27, p. 116.